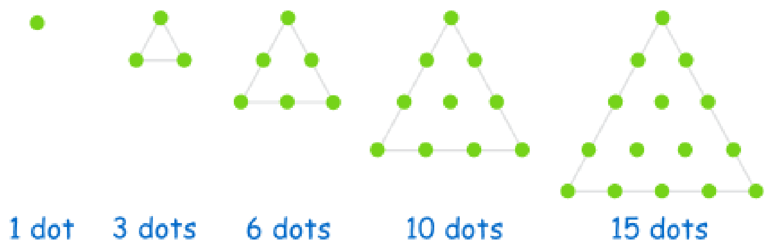
CH 43 – SEQUENCES & SERIES

❑ INTRODUCTION TO SEQUENCES



A ***sequence*** is a listing of numbers in a particular <u>order</u>. (Actually, a sequence can be a listing of almost anything you can imagine, like baseball games, but for this Algebra course we'll assume sequences of numbers.) For example, 1, 3, 8, 5 is a **finite** sequence of four terms, and is <u>different</u> from the finite sequence 3, 5, 1, 8 (even though each sequence consists of the same four numbers).

A sequence can also be **infinite**; an example would be

2, 4, 6, 8, 10, 12, . . .

which represents <u>all</u> the **even** natural numbers in increasing order.

> The three dots "**. . .**" is an English punctuation symbol called an *ellipsis,* and denotes an *omission* of something.

Since *order* is the key idea behind a sequence, a sequence has a first term, a second term, a third term, and so on. For instance, the sequence (notice the ellipsis)

5, 8, 11, 14, **. . .** , 29

is a finite sequence comprising nine terms:

5, 8, 11, 14, 17, 20, 23, 26, 29

The second term is 8, the fifth term is 17, and the ninth term is 29.

Our goal is to look at the first few terms of a sequence, and then try to determine the value of a term farther out in the sequence.

Homework

1. Determine whether the sequence is finite or infinite.

   a. 2, 3, 4, 5, . . . b. 5, 7, 9, . . . , 1001 c. 7, 7, 7, 7, . . .

2. a. What is the 7th term of the sequence 4, 8, 3, 0, 0, 9, 13, −1, π?

   b. Are the sequences 1, 2, 3, 4 and 4, 3, 2, 1 the same sequence?

❑ *EXAMPLES OF SEQUENCES*

<u>EXAMPLE 1:</u> **Consider the infinite sequence**

4, 8, 12, 16, 20, . . .

The 1st term is 4, the 2nd term is 8, and so on. What's the 10th term?

<u>Solution:</u> Here's an approach: Find the pattern among the existing terms, extend that pattern, and then predict the 10th term. Each term of the sequence appears to be four more than the preceding term. Continuing this pattern, it appears that the first 10 terms of the sequence are

4, 8, 12, 16, 20, 24, 28, 32, 36, 40.

Hence, **the 10th term is 40**, and we've accomplished our goal.

<u>EXAMPLE 2:</u> **Consider the sequence in Example 1 again: 4, 8, 12, 16, 20, . . . Find the 900th term.**

<u>Solution:</u> Would you like to list 900 terms to see what the 900th term is? I didn't think so! Let's attack this problem from a different perspective. Here's the idea in a nutshell: *Instead of*

seeing how each term comes from the ones before it, let's see how each term comes from its *position* *in the sequence.* We'll let ***n*** represent the position.

n	1	2	3	4	5
Term	4	8	12	16	20

This may not be obvious at first, but notice that each term is 4 times its position. For example, the term 12 is 4 times its position of 3. The term 20 is 4 times its position of 5. As a formula, we can write

$$n\text{th term} = 4n$$

Answering the original question is now a snap. The 900th term is given by the expression 4(900), which is **3,600**, and we're done.

EXAMPLE 3: **Find the 1,000th term of the sequence 4, 7, 10, 13, 16, . . .**

Solution: As in Example 2, let's construct a chart showing each term of the sequence as a function of its position:

n	1	2	3	4	5
Term	4	7	10	13	16

A little analysis (a phrase that means "it's not easy!") shows that each term is 1 more than 3 times its position. For example, the term 16 is 1 more than 3 times its position of 5. In general,

$$n\text{th term} = 3n + 1$$

Therefore, the 1,000th term is 3(1,000) + 1 = **3,001**.

EXAMPLE 4: **Find the 1,000th term of 1, 4, 9, 16, 25, 36, . . .**

Solution: Listing our data in a nice chart gives:

n	1	2	3	4	5	6
Term	1	4	9	16	25	36

This example doesn't have the same kind of formula that's in the previous examples, since the terms are not growing at a constant rate; the fact is, they're getting big very quickly. So let's look at it in a somewhat different way.

Notice that the term 25 is the square of its position $n = 5$. In fact, every term of the sequence is the *square* of its position (you verify this, for if there's just one place where our theory fails, then we have the wrong formula). We can write this as

$$\boldsymbol{n}\textbf{th term} = \boldsymbol{n^2}$$

And now it's as easy as π. The 1,000th term is $1{,}000^2$ = **1,000,000**.

EXAMPLE 5: **Find the 500th term of 2, 5, 10, 17, 26, 37, . . .**

Solution: Let's set up a chart.

n	1	2	3	4	5	6
Term	2	5	10	17	26	37

The trick here is to notice that each term of this sequence is one more than the square of the position. For instance, the term 26 is 1 more than the square of its position of 5. Thus, our formula here needs to be

$$\boldsymbol{n}\textbf{th term} = \boldsymbol{n^2 + 1}$$

The 500th term is therefore $500^2 + 1$ = **250,001**.

EXAMPLE 6: **Find the 25th term of 2, 4, 8, 16, 32, 64, 128, . . .**

Solution:

n	1	2	3	4	5	6	7
Term	2	4	8	16	32	64	128

This one is also kind of obscure. But notice that each term is twice the previous term. This leads us to the realization that each term of the sequence can be found by raising 2 to the power of its position. For instance, the 5th term is $2^5 = 32$. Check out the first term: $2^1 = 2$. Also, the 7th term, which is 128, is indeed 2^7. Our summary is therefore

$$\boldsymbol{n\text{th term} = 2^n}$$

The 25th term must be 2^{25} = **33,554,432**.

EXAMPLE 7: **Consider the infinite sequence**

–1, 1, 5, 13, 29, 61, 125, . . .

Find the 20th term.

Solution:

n	1	2	3	4	5	6	7
Term	–1	1	5	13	29	61	125

Think powers of 2 (as in the previous example). But this time notice that each of the terms of the sequence is <u>3 less than</u> 2 raised to the power of its position. As an example, the 6th term (61) is 3 less than 2^6 (64). In general,

$$\boldsymbol{n\text{th term} = 2^n - 3}$$

In conclusion, the 20th term is $2^{20} - 3$ = **1,048,573**.

EXAMPLE 8: **Find the 125th term of 1, 8, 27, 64, 125, 216, . . .**

Solution: Don't be ashamed if you don't see a pattern here. I wouldn't be able to see it either if I hadn't made up the problem myself. This is the sequence of perfect cubes:

$$1^3, 2^3, 3^3, 4^3, 5^3, 6^3, \ldots$$

Thus, the formula for the *n*th term is

$$\boldsymbol{n\text{th term} = n^3}$$

We therefore have a 125th term of $125^3 = $ **1,953,125**.

EXAMPLE 9: **Find the 60th term of 0, 7, 26, 63, 124, 215, . . .**

Solution: Comparing this sequence with the sequence of perfect cubes in Example 8, we see that each term is 1 less than a perfect cube, so the formula must be

$$\boldsymbol{n\text{th term} = n^3 - 1}$$

Letting $n = 60$, we calculate the 60th term to be **215,999**.

Homework

> "Life does not need to be changed. Only your intent and actions do."
>
> *Swami Rama*

3. Find the 2000th term of 7, 12, 17, 22, 27, . . .
4. Find the 100th term of −1, 2, 7, 14, 23, 34, . . .
5. Find the 25th term of 5, 7, 11, 19, 35, 67, . . .
6. Find the 57th term of −1, 6, 25, 62, 123, . . .
7. Find the 7354th term of 6, 9, 14, 21, 30, 41, 54, . . .
8. Find the 3000th term of 13, 17, 21, 25, 29, . . .

9. Find the 123rd term of 4, 11, 30, 67, 128, . . .
10. Find the 30th term of –3, –1, 3, 11, 27, 59, . . .
11. Find the 62nd term of 4, 7, 10, 13, 16, . . .
12. Find the 81st term of 4, 7, 12, 19, 28, 39, . . .
13. Find the 71st term of –2, 5, 24, 61, 122, 213, . . .
14. Find the 87th term of 0, 7, 26, 63, 124, 215, . . .
15. Find the 67th term of –3, 0, 5, 12, 21, 32, . . .
16. Find the 64th term of 2, 5, 10, 17, 26, 37, . . .

❑ *SERIES*

Series is a fancy term for *sum*, or summation. A **finite** series might be the summation

$$1 + 2 + 5 + 6 + 10 \qquad \text{(which has a sum of 24)}$$

Sometimes, if there's a nice pattern, a series can be written in "**sigma**" notation. The Greek capital letter sigma, Σ, is used to represent the summation (since sigma and sum both begin with the letter *s*). The following examples should explain how this notation is to be read and calculated.

EXAMPLE 10: **Evaluate:** $\sum_{k=2}^{5} (2k+1)$

Solution: The expression which determines the numbers we will add together is $2k + 1$. Below the sigma sign is the starting value of k, in this case 2; above the sigma sign is the ending value of k, in this case 5. So k starts at 2 and ends at 5, but what does k do in between? We agree that it goes up by one—in other words, k will go 2, 3, 4, and then 5. Check it out, remembering that the sigma sign, Σ, means ADD:

$$\sum_{k=2}^{5}(2k+1) = \underset{(k=2)}{(2\cdot\mathbf{2}+1)} + \underset{(k=3)}{(2\cdot\mathbf{3}+1)} + \underset{(k=4)}{(2\cdot\mathbf{4}+1)} + \underset{(k=5)}{(2\cdot\mathbf{5}+1)}$$

$$= 5 + 7 + 9 + 11$$

$$= \mathbf{32}$$

EXAMPLE 11: **Evaluate:** $\displaystyle\sum_{n=0}^{3}(n^2-n)$

Solution: Does it matter that Example 10 used the variable k and this example uses the variable n? Not at all — the variable used makes no difference in the final answer; it's just a placeholder. The values of n will be 0, 1, 2, and 3. For each value of n we evaluate the expression $n^2 - n$. Then we add up the results.

$$\sum_{n=0}^{3}(n^2-n) = \underset{(n=0)}{(\mathbf{0}^2-\mathbf{0})} + \underset{(n=1)}{(\mathbf{1}^2-\mathbf{1})} + \underset{(n=2)}{(\mathbf{2}^2-\mathbf{2})} + \underset{(n=3)}{(\mathbf{3}^2-\mathbf{3})}$$

$$= 0 + 0 + 2 + 6$$

$$= \mathbf{8}$$

Homework

Evaluate each series (find the sum):

17. $\displaystyle\sum_{k=3}^{5}(7k-1)$

18. $\displaystyle\sum_{n=2}^{5}(n^2+n)$

19. $\displaystyle\sum_{k=1}^{6}\frac{1}{2^k}$

20. $\displaystyle\sum_{j=0}^{4}3^j$

21. $\displaystyle\sum_{n=16}^{16}\tfrac{1}{2}\sqrt{n}$

22. $\displaystyle\sum_{k=3}^{5}\frac{1}{k}$

23. $\displaystyle\sum_{m=0}^{2}\frac{1}{m+1}$

24. $\displaystyle\sum_{t=-1}^{4}2t$

25. $\displaystyle\sum_{j=-2}^{2}j^3$

26. $\sum_{k=1}^{3} \frac{1}{k^2}$ 27. $\sum_{n=0}^{3} (n^2 - n + 1)$ 28. $\sum_{k=1}^{99} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

29. $\sum_{k=1}^{5} (-1)^k \cdot k^2$ 30. $\sum_{j=1}^{3} (-1)^{j+1} \cdot j$ 31. $\sum_{n=0}^{3} \frac{(-1)^n}{2n+1}$

Convert each sum to sigma notation – use *k* for the index variable, and always start with the value *k* = 0:

Example: $2 + 4 + 6 + 8 + 10 = \sum_{k=0}^{4} (2k + 2)$

32. $1 + 2 + 3 + 4 + 5 + 6$

33. $2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

34. $4 + 8 + 16 + 32 + 64 + 128$

35. $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$

36. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

37. $1 + 4 + 9 + 16 + 25 + 36 + 49$

38. $-2 + 4 - 8 + 16 - 32 + 64$

39. $3 - 5 + 7 - 9 + 11 - 13 + 15 - 17$

Practice Problems

40. Find the 32nd term of 5, 7, 11, 19, 35, 67, 131, . . .
41. Find the 200th term of 10, 13, 16, 19, . . .
42. Find the 100th term of 4, 7, 12, 19, 28, . . .
43. Find the 30th term of 1, 3, 7, 15, 31, 63, . . .
44. Find the 100th term of 3, 10, 29, 66, 127, . . .

45. $\sum_{n=2}^{5} (n^2 + n) =$ 46. $\sum_{k=1}^{6} \frac{1}{2^k} =$ 47. $\sum_{j=0}^{4} 3^j =$

48. $\sum_{n=16}^{16} \frac{1}{2}\sqrt{n} =$ 49. $\sum_{k=1}^{10} k =$ 50. $\sum_{n=0}^{4} 2^{n-2} =$

51. True/False:

a. The sequence 1, 2, 3 is the same sequence as 3, 2, 1.

b. It's possible that the terms of a sequence are all the same.

c. The 2,000th term of 20, 23, 26, 29, . . . is 6017.

d. The 100th term of 11, 14, 19, 26, 35, . . . is 10,000.

e. The 100th term of –9, –2, 17, 54, 115, . . is 999,990.

f. The 30th term of 10, 12, 16, 24, 40, 72, . . . is 1,073,741,832.

g. $\sum_{i=2}^{5} i^2 = 54$ h. $\sum_{j=0}^{3} 2^j = 16$

i. $\sum_{k=-1}^{1} k^3 = 3$ j. $\sum_{L=0}^{4} \sqrt{L} = 3 + \sqrt{2} + \sqrt{3}$

52. Convert to sigma notation—use ***k*** for the index variable starting at 0.

a. 8 + 11 + 14 + 17 + 20 + 23 b. $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$

c. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$ d. 8 + 27 + 64 + 125

Solutions

1. a. Infinite b. Finite c. Infinite

2. a. 13 b. No **3.** 10,002 **4.** 9,998 **5.** 33,554,435

6. 185,191 **7.** 54,081,321 **8.** 12,009 **9.** 1,860,870

10. 1,073,741,819 **11.** 187 **12.** 6,564 **13.** 357,908

14. 658,502 **15.** 4,485 **16.** 4,097 **17.** 81 **18.** 68

19. $\frac{63}{64}$ **20.** 121 **21.** 2 **22.** $\frac{47}{60}$ **23.** $\frac{11}{6}$ **24.** 18

25. 0 **26.** $\frac{49}{36}$ **27.** 12 **28.** $\frac{99}{100}$

29. –15 **30.** 2 **31.** $\frac{76}{105}$

32. $\sum_{k=0}^{5}(k+1)$ **33.** $\sum_{k=0}^{8}(k+2)$ **34.** $\sum_{k=0}^{5}2^{k+2}$ **35.** $\sum_{k=0}^{5}\frac{1}{k+5}$

36. Think powers of 2.

37. $\sum_{k=0}^{6}(k+1)^2$ **38.** $\sum_{k=0}^{5}(-1)^{k+1}2^{k+1}$ or $\sum_{k=0}^{5}(-2)^{k+1}$

39. $\sum_{k=0}^{7}(-1)^k(2k+3)$

40. 4,294,967,299

41. 607 **42.** 10,003 **43.** 1,073,741,823 **44.** 1,000,002

45. 68 **46.** $\frac{63}{64}$ **47.** 121 **48.** 2 **49.** 55 **50.** $\frac{31}{4}$

51. a. F b. T c. T d. F e. T
f. T g. T h. F i. F j. T

52. a. $\sum_{k=0}^{5}(3k+8)$ b. $\sum_{k=0}^{4}\frac{1}{2^{k+3}}$

c. $\sum_{k=0}^{5}\frac{(-1)^k}{(k+1)^2}$ d. $\sum_{k=0}^{3}(k+2)^3$

"I think and think for months and years.
Ninety-nine times, the conclusion is false.
The hundredth time I am right."

– Albert Einstein